Differentiation

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Literature

- A. C. Chiang and K. Wainwright, *Fundamental methods of mathematical economics*, Irwin/McGraw-Hill, Boston, MA, 2005.
- P. Hammond and K. Sydster, *Essential mathematics for economic analysis*, Financial Times Prentice Hall, Harlow, Munich, 2006.
- D. Stefanica, *A primer for the mathematics of financial engineering*, Financial Engineering Press, New York, NY, 2008.

Online Resources

- http://ocw.mit.edu/courses/audio-video-courses/#mathematics
  (Single Variable Calculus, Multivariable Calculus, Differential Equations, Linear Algebra)
- http://ocw.mit.edu/courses/audio-video-courses/#electrical-engineering-and-computer-science
  (Probabilistic Systems Analysis and Applied Probability)
- https://www.coursera.org/
  (plenty of courses on different subjects)

Motivating Example

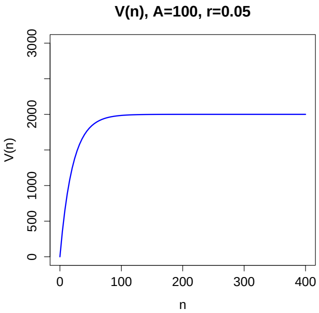
- **Annuity:** a financial product which you buy and that regularly pays you a fixed amount of money over a period of time
- Suppose, that you're offered to buy an annuity that pays $100 at the end of each year for the next 10 years. At an interest rate of 4%, how much are you willing to pay?
- To calculate the present value of the annuity, use the following formula: $V(n, r, A) = \frac{A}{r} * \left[1 - \frac{1}{(1+r)^n}\right]$, where A - the received amount, r - the interest rate and n - number of periods over which the annual payments are received.
- In our example $A = 100, r = 0.05, n = 10$. Therefore, $V(10, 0.05, 100) = 772.1735$.

What happens to $V(n)$ if we increase n?

Motivating Example

- $V(n, 0.05, 100) = \frac{100}{0.05} * \left[1 - \frac{1}{(1+0.05)^n}\right]$

  Graphically:

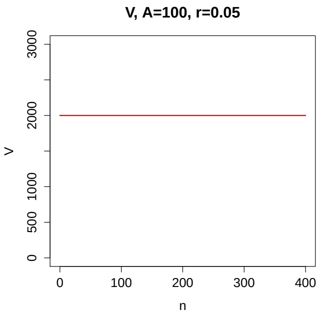


- What happens to $V(n)$ if we increase n?

  $\Leftrightarrow \lim_{n\to\infty} V(n) = \lim_{n\to\infty} \left(\frac{100}{0.05} * \left[1 - \frac{1}{(1+0.05)^n}\right]\right) = ?$

Motivating Example

- $\lim_{n\to\infty} V(n) = \lim_{n\to\infty} \left(\frac{A}{r} * \left[1 - \frac{1}{(1+r)^n} \right] \right) = \frac{A}{r}$
- $\lim_{n\to\infty} V(n) = \frac{100}{0.05} = 2000$
- **Graphically:**



Motivating Example

- $\lim_{n\to\infty} V(n) = \frac{A}{r}$ is an "easy limit".

What about:

- $\lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x} = ?$
- $\lim_{x\to 2} \frac{x^2+3x-10}{x-2} = ?$
- $\lim_{x\to 0} \frac{e^{2x}-1}{x} = ?$
- $\lim_{x\to\infty} \frac{ln(x)}{x} = ?$

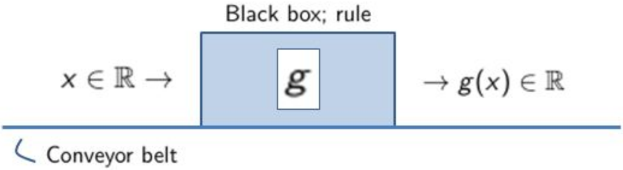
Rules of how to deal with limits are to be introduced.

Notation

- $\mathbb{N}$ - the set of natural numbers $(0, 1, 2, 3, \ldots)$
- $\mathbb{Z}$ - the set of integers $(0, \pm 1, \pm 2, \pm 3, \ldots)$
- $\mathbb{R}$ - the set of real numbers (any point of the real line)
- $\mathbb{R}_{++}$ - the set of positive real numbers
- $\in$ - in: indicator of set membership
- $\forall$ - for all
- $\exists$ - there exists
- $\rightarrow$ - goes to
- $g : \mathbb{R} \rightarrow \mathbb{R}$ - a real-valued function with a real-valued argument
- $/$ - not, e.g. $x = \sqrt{3}, x \notin \mathbb{Z}$
- $k! = k \cdot (k-1) \cdot (k-2) \cdot \ldots 2 \cdot 1$

Basic Definitions

- $g : \mathbb{R} \to \mathbb{R}$
- g - real-valued function with a real-valued argument



- g takes a real number as an argument and produces a real number as an output
- Notice that $g : \mathbb{R} \to \mathbb{R}$ doesn't mean that any real value can be obtained as an output (e.g. $g(x) = x^2$)

Basic Definitions

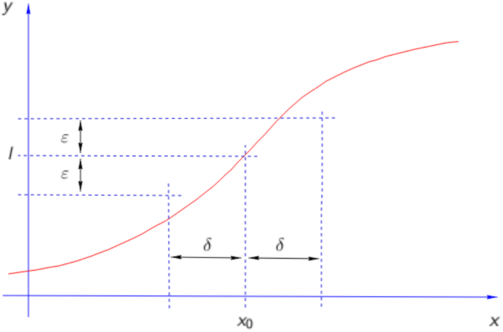
- The **limit** of $g(x)$ as $x \to x_0$ **exists** and is **finite** and **equals to** l if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|g(x) - l| < \varepsilon$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

- **Read as\Elaborately:** The **limit** of $g(x)$ as $x \to x_0$ (as argument x approaches a fixed value x_0) **exists** (notice that ∞ and $(-\infty)$ are also included into "**exists**") and is **finite** and **equals to** l if and only if for any $\varepsilon > 0$ (very small positive number ε) there exists $\delta > 0$ such that $|g(x) - l| < \varepsilon$ (we can make $g(x)$ very close to the limiting value l) for all $x \in (x_0 - \delta, x_0 + \delta)$ (for every x in the interval).

Basic Definitions

- **Mathematically:** $\lim_{x \to x_0} g(x) = l$ iff $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|g(x) - l| < \varepsilon$, $\forall\ |x - x_0| < \delta$.

- **Graphically:**



Basic Definitions

- $\lim_{x \to x_0} g(x) = \infty$ iff $\forall\ C > 0$, $\exists\ \delta > 0$ s.t. $g(x) > C$, $\forall |x - x_0| < \delta$.

- $\lim_{x \to x_0} g(x) = -\infty$ iff $\forall\ C < 0$, $\exists\ \delta > 0$ s.t. $g(x) < C$, $\forall |x - x_0| < \delta$.

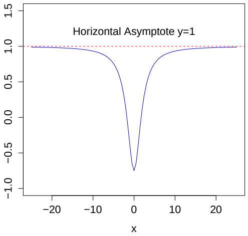
- $\lim_{x \to \infty} g(x) = l$ iff $\forall\ \varepsilon > 0$, $\exists\ b$ s.t. $|g(x) - l| < \varepsilon$, $\forall\ x > b$.

Basic Definitions: Horizontal Asymptotes

- What happens to $f(x) = \frac{x^2-3}{x^2+4}$ as the argument x goes to infinity?

  $$\Leftrightarrow \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-3}{x^2+4} = ?$$

- **Definition:** the line $y = L$ is called a **horizontal asymptote** if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ (notice that the L's in the definitions are not necessarily the same)
- **Graphically:**



Basic Definitions

- $\lim_{x \to x_0^+}$ - right-hand limit (i.e. $x \to x_0$ and $x > x_0$)

- $\lim_{x \to x_0^-}$ - left-hand limit (i.e. $x \to x_0$ and $x < x_0$)

- Example:
- $f(x) = \begin{cases} x+1, & x > 0 \\ -x+2, & x < 0 \end{cases}$
- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x+1) = 1$
- $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x+2) = 2$

Evaluating Limits

Suppose $c \in \mathbb{R}$ (i.e. c is a real-valued constant) and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist (i.e. need not to be finite but can't be of the form ${}^0/_0$ or ${}^\infty/_\infty$) then

- $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
- $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x)$
- $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$

Evaluating Limits

Let $c > 0$ be a positive constant then

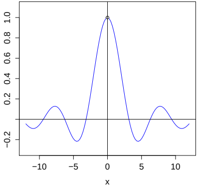
- $\lim_{x \to \infty} c^{\frac{1}{x}} = 1$
- $\lim_{x \to \infty} x^{\frac{1}{x}} = 1$

Let $k \in \mathbb{N} \setminus \{0\}$ and if $c > 0$ is a positive constant then

- $\lim_{k \to \infty} c^{\frac{1}{k}} = 1$
- $\lim_{k \to \infty} \frac{c^k}{k!} = 0$

Evaluating Limits: L'Hôpital's Rule

- Evaluate the following expression: $\lim_{x \to 0} \frac{sin(x)}{x}$.
- Remember the rules for evaluating limits.
  If we use the rule for the quotient of two functions, we will obtain the following expression: $\lim_{x \to 0} \frac{sin(x)}{x} = \frac{\lim_{x \to 0} sin(x)}{\lim_{x \to 0} 0} = \frac{0}{0}$
- Division by zero is not allowed. The function is, however, defined over the entire real line except for the point $x = 0$.
- **Graphically:**



- $\Rightarrow$ Use L'Hôpital's Rule!

Evaluating Limits: L'Hôpital's Rule

- Let x_0 be a real number (or $\pm\infty$) and let $f(x)$ and $g(x)$ be differentiable functions.
- **Rule:** Suppose $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ or $\pm\infty$. If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ exists and there is an interval (a, b) containing x_0 such that $g(x) \neq 0$ for all $x \in (a, b) \setminus x_0$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ exists and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$.

Evaluating Limits: L'Hôpital's Rule

Examples:

- $\lim_{x\to 0} \frac{sin(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx} sin(x)}{\frac{d}{dx} x} = \lim_{x\to 0} cos(x) = 1$
- $\lim_{x\to\infty} \frac{ln(x)}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx} ln(x)}{\frac{d}{dx} x} = \lim_{x\to\infty} \frac{1}{x} = 0$
- $\lim_{x\to 0} \frac{e^{e\cdot x} - 1}{e\cdot x} = \lim_{x\to 0} \frac{\frac{d}{dx}\left(e^{e\cdot x} - 1\right)}{\frac{d}{dx}\left(e\cdot x\right)} = \lim_{x\to 0} \frac{e\cdot e^{e\cdot x}}{e} = 1$
- $\lim_{x\to\infty} \frac{x^3 - 2}{x^2 + 3} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(x^3 - 2\right)}{\frac{d}{dx}\left(x^2 + 3\right)} = \lim_{x\to\infty} \frac{3x^2}{2x} =$
  $\lim_{x\to\infty} \frac{\frac{d}{dx}\left(3x^2\right)}{\frac{d}{dx}\left(2x\right)} = \lim_{x\to\infty} \frac{6x}{2} = \infty$

Types of Limits

1. "Easy" Limits:

   e.g. $\lim_{x \to 4} \frac{x+3}{x^2+1} = \frac{4+3}{4^2+1} = \frac{7}{17}$

2. "Harder" Limits:

   e.g. $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ results into the uncertainty of the $\frac{0}{0}$-type.
   Some further "manipulations" are needed.

Types of Limits

Example:

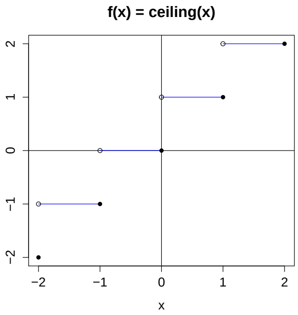
- $$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x} =$$
  $$\lim_{x\to\infty} \frac{1}{\sqrt{x^2-4x+1}-x} * \frac{\sqrt{x^2-4x+1}+x}{\sqrt{x^2-4x+1}+x} =$$
  $$\lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{x^2-4x+1-x^2} = \lim_{x\to\infty} \frac{\sqrt{x^2-4x+1}+x}{1-4x} * \frac{\frac{1}{x}}{\frac{1}{x}} =$$
  $$\lim_{x\to\infty} \frac{\sqrt{1-\frac{4}{x}-\frac{1}{x^2}}+1}{\frac{1}{x}-4} = \frac{2}{-4} = -\frac{1}{2}$$

Continuity

- **Definition:** a function $f(x)$ is continuous at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.
- The definition of continuity implies that:
  1. $\lim_{x \to x_0} f(x)$ exists (i.e. $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist and $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$)
  2. $f(x_0)$ is defined
  3. $\lim_{x \to x_0} f(x) = f(x_0)$
- **Definition:** a function $f(x)$ is right-continuous at x_0 if $\lim_{x \to x_0^+} f(x) = f(x_0)$.
- **Definition:** a function $f(x)$ is left-continuous at x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$.

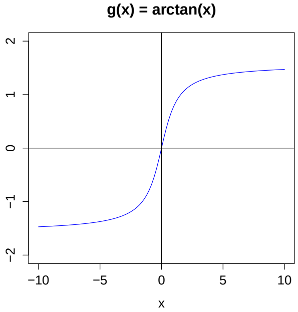
Continuity

- **Definition:** a function $f(x)$ is continuous on an interval (a, b) if it is continuous at every $x \in (a, b)$.
- Example: left-continuous function $f(x) = \lceil x \rceil$.



Continuity

- Example: function $g(x) = arctan(x)$ is continuous throughout the entire real line $\mathbb{R}$.



- **Intuitively:** a function is continuous if it can be drawn without taking the arm away from the paper.

Continuity Theorems

- Let f and g be continuous functions at x_0 and let $c \in \mathbb{R}$, then:
  1. $f + g$
  2. $f - g$
  3. $c \cdot f$
  4. $f \cdot g$
  5. $\frac{f}{g}$ (if $g(x_0) \neq 0$)

  are also continuous functions at x_0.

- If g is continuous at x_0 and f is continuous at $g(x_0)$ then $f \circ g(x) = f\left(g(x)\right)$ is also continuous at x_0.

Continuity and Frequently Used Functions.

- The following functions are continuous on their domains:
  - Polynomials ($\mathbb{D} = \mathbb{R} = (-\infty, \infty)$)
  - Roots functions ($\mathbb{D} = \mathbb{R}_+ = [0, \infty)$)
  - Logarithmic functions ($\mathbb{D} = \mathbb{R}_{++} = (0, \infty)$)
  - Exponential functions ($\mathbb{D} = \mathbb{R} = (-\infty, \infty)$)
- Example. Where is the function $f(x) = \frac{ln(x)}{x^2-1}$ continuous?
  1. The numerator: $g(x) = ln(x)$ is continuous on $\mathbb{D} = \mathbb{R}_{++} = (0, \infty)$.
  2. The denominator: $h(x) = x^2 - 1$ is continuous on $\mathbb{D} = \mathbb{R} = (-\infty, \infty)$.
  3. The entire function $f(x)$ is a quotient of two other functions. We should, therefore, exclude the values of x that turn the denominator of the function into 0 ($x^2 - 1 = (x-1) \cdot (x+1) = 0$ if $x = \pm 1$).
  4. Taking the intersection of the determined intervals and excluding the points $x = \pm 1$, we conclude that $f(x)$ is continuous on $(0, 1) \cup (1, \infty)$.

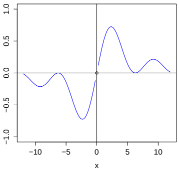
Examples of Discontinuities

Some functions are, however, discontinuous.

Examples of discontinuities:

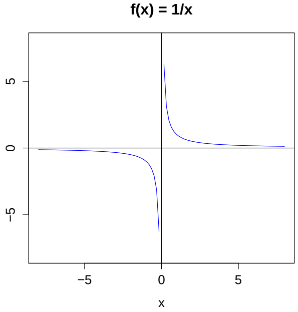
- **Example 1 (Jump Discontinuities):** $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist but $\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x)$ (see slide 15).
- **Example 2:** $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist and $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$ but $f(x_0)$ is not defined.

  Example: function $f(x) = \frac{1 - cos(x)}{x}$ is not defined at $x_0 = 0$.



Examples of Discontinuities

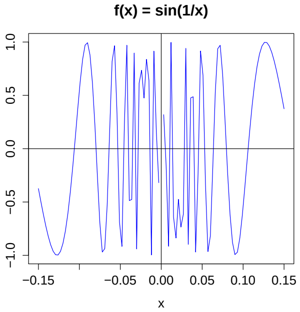
- **Example 3 (Infinite Discontinuities):**
  As an example consider the function $f(x) = \frac{1}{x}$.
  Graphically:



The $\lim_{x \to 0^-} f(x) = -\infty$ but $\lim_{x \to 0^+} f(x) = +\infty$.

Examples of Discontinuities

- **Example 4:** Consider the function $f(x) = sin(\frac{1}{x})$ as $x \to 0$.
  Graphically:



Neither left, not right limit exists!

Continuity: Theorem (Differentiability $\Rightarrow$ Continuity)

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- **Theorem (Differentiability $\Rightarrow$ Continuity):** if $f(x)$ is differentiable at x_0 then $f(x)$ is continuous at x_0.
  Proof:
  1. We can rewrite the definition of continuity as $\lim_{x \to x_0} f(x) - f(x_0) \stackrel{?}{=} 0$. This is what we need to show.
  2. $\lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0$ ■

- Notice that the converse statement is not true, i.e. Differentiability $\not\Leftarrow$ Continuity.

Continuity: Theorem (Differentiability $\Rightarrow$ Continuity)

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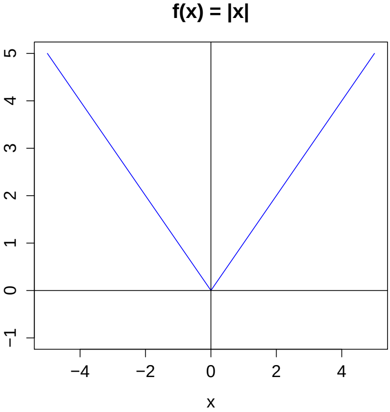
- Example: the function $f(x) = |x|$ is continuous on $\mathbb{D} = \mathbb{R}$ but is not differentiable at $x_0 = 0$.
  Graphically:



Differentiation

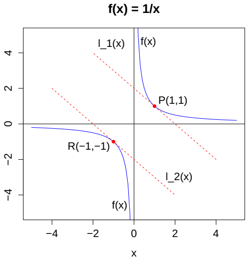
The goal of this lecture is to provide theoretical knowledge to answer the following two questions:

1. What is a derivative?
   - geometrical interpretation
   - physical interpretation
2. How to differentiate any function you know?

   $\frac{d}{dx}\left(e^{\frac{ln(x)}{arctan(x)}}\right) = ?$

Derivatives. Geometrical Interpretation.

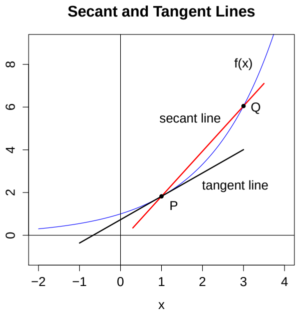
- Suppose that we need to find an equation for the tangent line $y = l(x)$ to the function $y = f(x)$ at some point $P(x_0, y_0)$.
  Graphically:



- **Definition:** a line $y = l(x)$ is tangent to the curve $f(x)$ at a point $P(x_0, y_0)$ if $\exists\ \delta > 0$ s.t.:
  - $f(x) > l(x)$ on $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ or
  - $f(x) < l(x)$ on $(x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ and
  - $f(x_0) = l(x_0)$.

Derivatives. Geometrical Interpretation.

- **Definition:** a secant line of a curve is a line that intersects two points on the curve.
  Graphically:



- Tangent line = limit of the secant lines PQ as $Q \to P$ (given that P stays fixed).

Derivatives. Geometrical Interpretation.

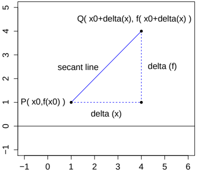
- The slope of the curve $f(x)$ equals to the slope of the line $l(x)$ at the point x_0.
- Back to the original task: determining the equation for the tangent line.
  A line $l(x)$ passing through $P(x_0, y_0)$ is determined by the following expression:

$$y - y_0 = m * (x - x_0)$$

  .
  Thus, to determine the equation for the tangent line we need:
  1. point $P(x_0, y_0 = f(x_0))$ and
  2. slope $m = f'(x_0)$ (the only calculus part).

Derivatives. Geometrical Interpretation.

- **Definition:** the derivative of $f(x)$ at x_0, denoted by $f'(x_0)$, is the slope of the tangent line to $y = f(x)$ at $P(x_0, y_0)$.
- **Finding the slope of the tangent line:**



- $$\underbrace{\frac{\Delta f}{\Delta x}}_{\text{slope of the secant line}} \quad \xrightarrow{\Delta x \to 0} \quad \underbrace{m = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}}_{\text{slope of the tangent line}}$$

Derivatives. Geometrical Interpretation.

- **Summing things up:**
  the slope of the tangent line at $P(x_0, y_0)$ is given by:

$$\boxed{f'(x_0) = m = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}}$$

- **Definition:** a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at a point $x_0 \in \mathbb{R}$ if $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ exists (notice that the limit needs not to be finite).
- **Definition:** a function $f(x)$ is differentiable at an open interval (a, b) if it is differentiable at every point $x \in (a, b)$.

Derivatives. Geometrical Interpretation.

Example: using the definition introduced above derive the derivative of the function $f(x) = \frac{1}{x}$ at x_0.

1. Construct the difference quotient $\frac{\Delta f}{\Delta x}$ first:

$$\frac{\Delta f}{\Delta x} = \frac{\frac{1}{x_0+\Delta x} - \frac{1}{x_0}}{\Delta x} = \frac{1}{\Delta x} \cdot \left(\frac{x_0 - (x_0+\Delta x)}{(x_0+\Delta x) \cdot x_0} \right) =$$
$$\frac{1}{\Delta x} \cdot \left(\frac{-\Delta x}{(x_0+\Delta x) \cdot x_0} \right) = \frac{1}{(x_0+\Delta x) \cdot x_0}.$$

2. Consider what happens as Δx approaches 0:

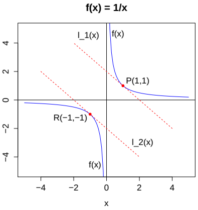
$$\frac{\Delta f}{\Delta x} = \frac{1}{(x_0+\Delta x) \cdot x_0} \xrightarrow{\Delta x \to 0} -\frac{1}{x_0^2}.$$

$$\boxed{f'(x_0) = -\frac{1}{x_0^2}}$$

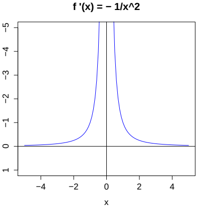
Derivatives. Geometrical Interpretation.

Graphically:

- Function $f(x) = \frac{1}{x}$:



- Derivative $f'(x) = -\frac{1}{x^2}$:



Derivatives. Geometrical Interpretation.

- A quick check-up of the consistency of the obtained results:
  1. The expression $f'(x) = -\frac{1}{x^2}$ is always negative which corresponds to negative slopes of the tangent lines to $f(x) = \frac{1}{x}$ at any point of the domain.
  2. As x goes to infinity the slope of the tangent lines becomes less and less steep which corresponds to $\lim_{x \to \infty} f'(x) = 0$.
- **Remark on notation:**
  notice that $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. Thus, writing that $\lim_{x \to 0} f(x) = \infty$ is not sloppy but simply wrong!
  However, $\lim_{x \to 0^+} -\frac{1}{x^2} = -\infty$ and $\lim_{x \to 0^-} -\frac{1}{x^2} = -\infty$. In this case saying that $\lim_{x \to 0} -\frac{1}{x^2} = -\infty$ is correct.

Derivatives. More Notation.

- $y = f(x), \Delta y = \Delta f$;
  $$\underbrace{f'}_{\text{Newton's notation}} = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f}_{\text{Leibniz's notation}}.$$
- Higher Derivatives:
  If $u = u(x)$ is an n-time differentiable function of x then:
  - $u'(x) = \frac{du}{dx} = Dx$ is also a function of x and is referred to as the first derivative of $u(x)$.
  - $(u'(x))' = u''(x) = \frac{d}{dx}\frac{du}{dx} = \frac{d^2}{dx^2}u = D^2u$ is also a function of x and is referred to as the second derivative of $u(x)$.

    $\vdots$
  - $\left(u^{(n-1)}(x)\right)' = u^{(n)}(x) = \frac{d}{dx}\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^n}{dx^n}u = D^nu$ is also a function of x and is referred to as the n^{th} derivative of $u(x)$.

Derivatives.

Example: using the definition introduced above derive the derivative of the function $f(x) = x^n$ for $n \in \mathbb{N}$.

1. Construct the difference quotient $\frac{\Delta f}{\Delta x}$ first:

$$\frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x} = \frac{1}{\Delta x} \cdot \left(x^n + nx^{n-1}\Delta x + \mathbb{O}\left((\Delta x)^2\right) - x^n\right) =$$

$$\frac{1}{\Delta x} \cdot \left(nx^{n-1}\Delta x + \mathbb{O}\left((\Delta x)^2\right)\right) = nx^{n-1} + \mathbb{O}(\Delta x).$$

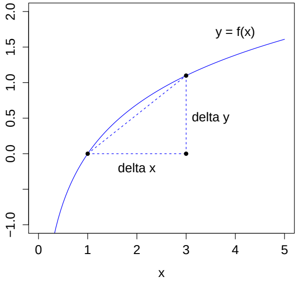
2. Consider what happens as Δx approaches 0:

$$\frac{\Delta f}{\Delta x} = nx^{n-1} + \mathbb{O}(\Delta x) \xrightarrow{\Delta x \to 0} nx^{n-1}.$$

$$\boxed{f'(x) = \frac{d}{dx}x^n = nx^{n-1}, n \in \mathbb{N}}$$

Derivatives. Physical Interpretation.

- Consider the following graph:



- $\frac{\Delta y}{\Delta x}$ - relative or average rate of change

  $\frac{\Delta y}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{dy}{dx}$ - instantaneous rate of change

Derivatives. Physical Interpretation.

Example (Pumpkin Drop):

- Suppose you're participating in a contest dedicated to the Halloween Celebration. The goal of the contest is to throw a pumpkin from the top of KG II as precisely on the mark on the nearby lawn as possible.

- Assuming that the equation $h(t) = 80 - 5t^2$ describes the coordinate of the vertical position of the pumpkin, calculate:
  1. the average velocity of the pumpkin assuming it was falling for $t = 4$ seconds;
  2. the instanteneous velocity of the pumpkin at $t = 4$.

Derivatives. Physical Interpretation.

Solution:

- The average velocity of the pumpkin throughout the time interval (t_0, t_1) is given by $\frac{\Delta h}{\Delta t} = \frac{h_1 - h_0}{t_1 - t_0} = \frac{0-80}{4-0} = -20$ m/s.

- The instanteneous velocity of the pumpkin at $t = 4$ is given by $\frac{d}{dt}h = \frac{d}{dt}(80 - 5t^2) = 0 - 10t$. At $t = 4$ the instanteneous velocity equals to $\frac{d}{dt}h\Big|_{t=4} = (0 - 10t)\big|_{t=4} = -40$ m/s.

Derivatives: Frequently Used Rules. Product Rule.

The Product Rule allows to take derivatives of the product of functions for which derivatives exist.

E.g. $\frac{d}{dx}(x^n sin(x)) = ?$

Product Rule: suppose $u(x)$ and $v(x)$ are differentiable functions, then $\boxed{(uv)' = u'v + uv'}$.

Derivation of the rule:

- Consider the change in the functional value first:
  $\Delta(uv) = u(x + \Delta x) \cdot v(x + \Delta x) - u(x) \cdot v(x) =$

  $(u(x + \Delta x) - u(x)) \cdot v(x + \Delta x) + u(x) \cdot v(x + \Delta x) - u(x) \cdot v(x) =$

  $(u(x + \Delta x) - u(x)) \cdot v(x + \Delta x) + u(x) \cdot (v(x + \Delta x) - v(x)) =$

  $\Delta u \cdot v(x + \Delta x) + u(x) \cdot \Delta v.$

Derivatives: Frequently Used Rules. Product Rule.

- Construct the difference quotient and consider what happens if $x \to 0$:

$$\frac{\Delta(uv)}{\Delta x} = \frac{\Delta u \cdot v(x+\Delta x) + u(x) \cdot \Delta v}{\Delta x} =$$

$$\frac{\Delta u}{\Delta x} \cdot v(x+\Delta x) + u(x) \cdot \frac{\Delta v}{\Delta x} \xrightarrow{\Delta x \to 0} \frac{du}{dx} v(x) + u(x) \frac{dv}{dx}. \quad \blacksquare$$

$$\boxed{\frac{d}{dx}(uv) = \frac{du}{dx} v(x) + u(x) \frac{dv}{dx}}$$

Derivatives: Frequently Used Rules. Quotient Rule.

The Quotient Rule allows to take derivatives of the quotient of two functions for which derivatives exist.

E.g. $\frac{d}{dx}\left(\frac{1}{x^n}\right) = ?$

Quotient Rule: suppose $u(x)$ and $v(x)$ are differentiable functions, then $\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$.

Derivation of the rule:

- Consider the change in the functional value first:

$$\Delta\left(\frac{u}{v}\right) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} = \frac{uv+(\Delta u)v-uv-(\Delta v)u}{(v+\Delta v)v} = \frac{v\cdot\Delta u-u\cdot\Delta v}{(v+\Delta v)\cdot v} \quad .$$

Derivatives: Frequently Used Rules. Quotient Rule.

- Construct the difference quotient and consider what happens if $x \to 0$:

$$\frac{\Delta\left(\frac{u}{v}\right)}{\Delta x} = \frac{\frac{\Delta u}{\Delta x} v - u \frac{\Delta v}{\Delta x}}{(v+\Delta v)v} \xrightarrow{\Delta x \to 0} \frac{\frac{du}{dx} v - \frac{dv}{dx} u}{v \cdot v}. \quad \blacksquare$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} v - \frac{dv}{dx} u}{v \cdot v}}$$

Derivatives: Frequently Used Rules. Chain Rule.

The Chain Rule allows to take derivatives of composite functions.

Chain Rule: if $f(x)$ and $g(x)$ are differentiable functions then the composite function $(g \circ f)(x) = g(f(x))$ is also differentiable and

$$\boxed{((g \circ f)(x))' = (g(f(x))' = g'(f(x)) \cdot f'(x)} \quad .$$

Example:

$$\frac{d}{dt}(sin(t))^{10} = \underbrace{10(sin(t))^9}_{\text{derivative of the outer function}} \cdot \underbrace{cos(t)}_{\text{derivative of the inner function}} .$$

Derivatives: Chain Rule and Substitution Method.

Substitution Method (Leibniz's Notation):
Suppose $f(x)$ and $g(x)$ are differentiable functions. Consider a composite function $g(f(x))$ and let $u = f(x)$, then $g(f(x)) = g(u)$

and $\boxed{\dfrac{d}{dx}g(u) = \dfrac{dg(u)}{du} \cdot \dfrac{du(u)}{dx}}$.

Example: consider the function $g(x) = sin(x^2)$. Find $\frac{d}{dx}sin(x^2)$ using the substitution method.
Solution:

- Let $f(x) = x^2 = u$, then $g(f(x)) = sin(x^2) = g(u)$.
- We can then find the derivative using the substitution method:

$$\tfrac{d}{dx}g(u) = \tfrac{d}{dx}sin(u) = \tfrac{dsin(u)}{du} \cdot \tfrac{du}{dx} = cos(u) \cdot 2x = cos(x^2) \cdot 2x \ .$$

Implicit Differentiation

- Consider the function $y = x^{\frac{m}{n}}$ with $m, n \in \mathbb{N} \setminus \{0\}$. Our goal is to find $\frac{d}{dx}x^{\frac{m}{n}}$.
- The approach will be to treat y as an **implicit** function of x. Therefore, $y = x^{\frac{m}{n}} \quad \Rightarrow \quad y^n = x^m$.

$$\underbrace{\frac{d}{dx}y^n}_{\text{chain rule}} = \tfrac{d}{dx}x^m \quad \Rightarrow \quad \left(\tfrac{d}{dy}y^n\right) \cdot \tfrac{dy}{dx} = mx^{m-1}$$

$$\Rightarrow ny^{n-1}\tfrac{dy}{dx} = mx^{m-1}$$

$$\Rightarrow \boxed{\frac{d}{dx}x^{\frac{m}{n}} = \frac{m}{n} \cdot x^{\frac{m}{n}-1}}$$

Implicit Differentiation

Example: y is defined by $y^4 + xy^2 - 2 = 0$ as an implicit function of x. Find the expression for $\frac{dy}{dx}$.

- **Explicit Solution:**

  $y^2 = \frac{-x \pm \sqrt{x^2+8}}{2}$

  $\Rightarrow y = \pm\sqrt{\frac{-x \pm \sqrt{x^2+8}}{2}}$
  Taking the derivative of this expression is very unpleasant.
- **Implicit Solution:** leave the expression as it is and differentiate the both sides.
  $\frac{d}{dx}(y^4 + xy^2 - 2 = 0)$

  $4y^3y' + y^2 + 2xyy' = 0$

  $(4y^3 + 2xy)y' = -y^2 \quad \Rightarrow \quad y' = \frac{-y^2}{4y^3+2xy}$.