Functions

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Basic Definitions

A **function** of a real variable x with **domain** D is a rule that asigns a **unique** real number to each number $x \in D$.
As x varies over the whole domain, the set of all possible resulting values $f(x)$ is called the **range** of f, that is:

$$B = \{y \in \mathbb{R} | y = f(x), x \in D\}$$

$$\begin{aligned} f : D &\to B \subset \mathbb{R} \\ x &\mapsto f(x) \end{aligned}$$

Basic Definitions

- The variable x is often called the **independent variable**, or the **argument** of f.
- One often defines $y = f(x)$. In this case y is refered to as the **dependent variable**.
- Function f is said to be **one-to-one** function in $A \subset D$ if f never has the same value for any two different points in A. This is equivalent to:
  f is a **one-to-one** function $\Leftrightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Basic Definitions

Let f be a function with domain D and range B. If and only if f is a one-to-one function, it has an **inverse function** f^{-1} with domain B and range D. For each $y \in B$, the value $f^{-1}(y)$ is the unique number $x \in D$ such that $f(x) = y$

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

Example

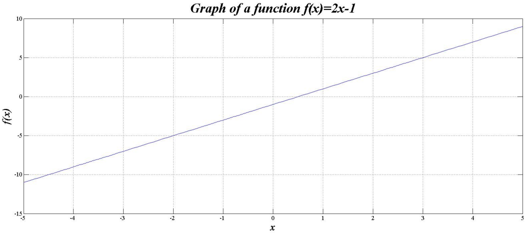
$$f : [0, 2] \to [0, 4]$$

$$f(x) = x^2 \Longrightarrow f^{-1} = \sqrt{x}$$

Note that $f^{-1} \neq \dfrac{1}{f}$

Graph of a Real Function

For a given real function $f(x)$, its grapgh is the graphical representation of the set $\{(x, f(x)) : x \in D\}$



Monotonic functions

Let f be a function defined on an interval $I \subset \mathbb{R}$ and x_1 and x_2 be two numbers in I. We say that:

- if $f(x_2) \geq f(x_1)$ whenever $x_2 \geq x_1$, then f is a **monotonic increasing** on I.
- if $f(x_2) > f(x_1)$ whenever $x_2 \geq x_1$, then f is **strictly monotonic increasing** on I.
- if $f(x_2) \leq f(x_1)$ whenever $x_2 \geq x_1$, then f is a **monotonic decreasing** on I.
- if $f(x_2) < f(x_1)$ whenever $x_2 \geq x_1$, then f is **strictly monotonic decreasing** on I.

Even (symmetric) and Odd (asymmetrical) Functions

- An univariate function $f(x)$ is said to be **even** if $f(x) = f(-x)$ Geometrically, such functions are symmetric about the y-axis.
- An inivariate function $f(x)$ is said to be **odd** if $f(x) = -f(-x)$.
  Geometrically, such functions are asymmetric about the y-axis.

Examples

- $f(x) = x^2$, $g(z) = z^4$, constant function, standard normal distribution function,...
- $f(x) = x$, $f(x) = x^3$,...

Linear Functions

A linear function is a function of the following form:

$$f(x) = ax + b$$

where $a, b \in \mathbb{R}$
a is the **slope** and b is the **intercept**

Examples

- $f(x) = 2x - 5$
- $g(z) = \frac{2}{5}z + 3.78$
- $g(x) = -18x + k, k \in \mathbb{R}$

Quadratic Functions

A quadratic function is a function of the following form:

$$f(x) = ax^2 + bx + c$$

where $a, b, c \in \mathbb{R}$

Examples

- $f(x) = x^2 + 10x - 3$
- $f(x) = -5x^2 + 4x$

Graph of a quadratic function is parabola. The shape of the parabola is defined by the coefficients a, b, and c.

Polynomials

Linear and quadratic functions are just examples of a more general class of polynomial function, which are the functions of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = \sum_{k=0}^{n} a_k x^k$$

where $a_k \in \mathbb{R}, k = 0, 1, 2, ..., n$. n is refered to as the **order** or **degree** of the given polynomial. a_0 is sometimes called the free coefficient.

Examples

- $f(x) = 20x^7 + 10x - 3$
- $f(x) = -5x^{125} + 4x^{14}$

Rational Functions

A **rational function** is a function that can be represented as a fraction of two polynomial functions.

$$R(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

Examples

- $G(z) = \frac{1+25z^{23}}{-98+38z^3-0.5z^{123}}$
- $I(x) = \frac{\sum_{i=1}^{k} a_k x^k}{1-x}$

Power Functions

A general **power function** is defined as:

$$f(x) = Ax^r$$

where A and r are constants. Note that if $r < 0$ one has to exclude $x = 0$ from the domain. power function

Example

- $g(y) = \frac{1}{4}y^4$
- $f(z) = 2z^{-\frac{1}{2}}$

Reminder: Operations with powers

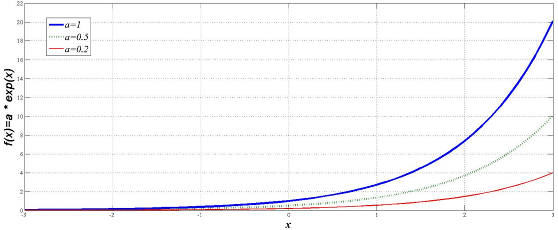
- $x^a \times x^b = x^{a+b}$
- $x^{-b} = \frac{1}{x^b}$
- $\frac{x^a}{x^b} = x^{a-b}$
- $x^{\frac{1}{a}} = \sqrt[a]{x}$
- $x^{\frac{b}{c}} = \sqrt[c]{x^b}$
- $x^a \times y^a = (xy)^a$

The Natural Exponential Function

The natural exponent function is the function

$$f(x) = ae^x$$

where $a \in \mathbb{R}$ and $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.718281828459045...$



Operations with exponents

- $e^x \times e^y = e^{x+y}$
- $e^{x-y} = \dfrac{e^x}{e^y}$
- $\prod_{i=1}^{n} e^{x_i} = e^{\sum_{i=1}^{n} x_i}$

Logarithmic Function (natural logarithm)

For each positive number x, the number $\ln(x)$ is defined by $e^{lnx} = x$.
Or in other words, $u = \ln(x)$ is the solution of the equation $e^u = x$

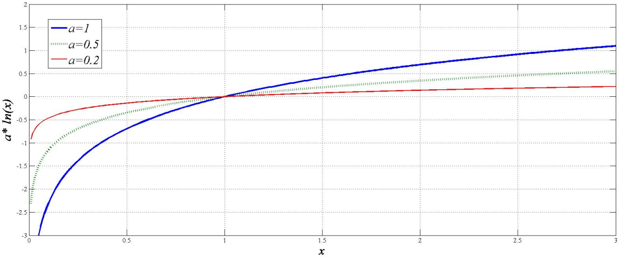
$$g(x) = \ln(x)$$

where $x > 0$.

Operations with logarithms

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$
- $\ln(x^a) = a\ln(x)$

Graphical representation:



So far we have mostly consiedered functions of a single variable, however one must often work with functions of multiple variables.

$$f : D \to \mathbb{R}$$

where the domain $D \in \mathbb{R}^n$, thus

$$x = (x_1, x_2, ..., x_n) \mapsto f(x)$$

A real function of multiple variables is usually defined as a combination of a single variable real functions.

Examples

- $f(x, y) = \log(x + y^2)$
- $g(x, y, z) = ye^x\sqrt{25z}$
- $u(x_1, x_2, x_3) = \pi_1 x_1 + \pi_2 x_2 + \pi_3 x_3$, where $\pi_1, \pi_2, \pi_3 \in \mathbb{R}$

Composition of Functions

Function composition is the application of one function to the results of another. Given two real functions $f(x) : \mathbb{R} \to \mathbb{R}$ and $g(y) : \mathbb{R} \to \mathbb{R}$ we define $f \circ g : \mathbb{R} \to \mathbb{R}$ as

$$f \circ g(x) = f(g(x))$$

Example

$$f(x) = 2x + 5, g(x) = 5e^{2x}$$
$$\Rightarrow g \circ f(x) = 5e^{4x+10} \text{ and } f \circ g(x) = 10e^{2x} + 5$$

Note that usually $g \circ f(x) \neq f \circ g(x)$!

Exercise

$f(x) = 2x^2 + 5, g(x) = e^{x-1}, h(x) = \sqrt{x}$ find $g \circ f \circ h(x)$

Compound Functions

Up to now we have seen functions which were defined by the same rule for all the values of the domain. However one can define a function by a different rule for each of a number of **disjoint** parts of the domain.

Example 1

$$f(x) = \begin{cases} x^2, & x \in [0,2] \\ 4, & x > 2 \\ 0, & otherwise \end{cases}$$

Example 2

A step function

$$g(y) = k, \ y \in [k, k+1) \text{ for } k = 0, 1, 2, ...$$

Some Useful Formulas

- $\sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$

For $x \in (-1, 1)$

- $\sum_{i=1}^{N} x^i = \dfrac{1 - x^{N+1}}{1 - x}$
- $\sum_{i=1}^{\infty} x^i = \dfrac{1}{1 - x}$
- **Newton's binomial formula**
  - $(a+b)^2 = a^2 + 2ab + b^2$
  - $(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

    $\vdots$
  - $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$ where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$