Integration

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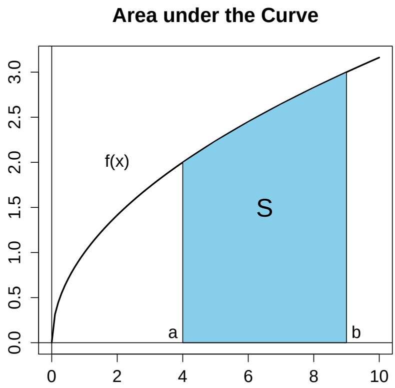
The Area Problem

- Consider a function $f(x)$ which is continuous on an interval $[a, b]$. Our goal today is to find the area of of the region S under $f(x)$ between a and b.

Graphically:



The Area Problem

To calculate the area of the region S we will use the lower and upper bounds of the area under the curve.

To calculate the **lower bound** of the area under the curve, we need to sum the areas of the rectangles calculated from the left (i.e. each rectangle is below the curve).

To calculate the **upper bound** of the area under the curve, we need to sum the areas of the rectangles calculated from the right (i.e each rectangle is above the graph).

The Area Problem

The calculation involves the following steps:

1. Create a partition P_n of the interval $[a, b]$ (n indicates the number of columns/intervals the interval $[a, b]$ is split into; here $n = 4$).
2. Use rectangles to calculate the lower and upper bounds of the area S.
3. Consider $\lim_{n\to\infty} L(P_n, f(x))$ and $\lim_{n\to\infty} U(P_n, f(x))$.

Notice that once you divide the interval into the larger number of segments, the lower bound of the area increases, whereas the upper bound of the area decreases. This is the reason behind considering $\lim_{n\to\infty} L(P_n, f(x))$ and $\lim_{n\to\infty} U(P_n, f(x))$.

The Area Problem

If $\lim_{n\to\infty} L(P_n, f(x)) = \lim_{n\to\infty} U(P_n, f(x))$, then the **definite integral** $f(x)$ from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} L(P_n, f(x)) = \lim_{n\to\infty} U(P_n, f(x)) = A \ ,$$

where A is the area of the shaded region S.

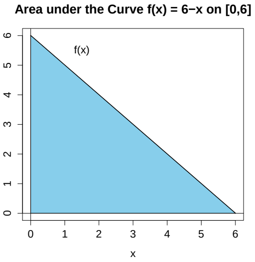
A function $f(x)$ is **integrable** if and only if

$$\int_a^b f(x)dx = \lim_{n\to\infty} L(P_n, f(x)) = \lim_{n\to\infty} U(P_n, f(x)) \ .$$

The Area Problem

Example: use the area interpretation of a definite integral to calculate the area under the curve given by $f(x) = 6 - x$ on the interval $[0, 6]$.

- What we need to calculate is the area of the shaded region under the curve.



- **Mathematically:** $\int_0^6 (6 - x)dx = \frac{1}{2} \cdot 6 \cdot 6 = 18$.

Properties of Definite Integrals

- $\int_b^a f(x)dx = -\int_a^b f(x)dx$
- $\int_a^a f(x)dx = 0$
  42mm]
- $\int_a^b c\ dx = c \cdot (b-a)$, for $c \in \mathbb{R}, b > a$
- $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ (linearity property)
- $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$, for $c \in \mathbb{R}$ (scaling property)

Properties of Definite Integrals

- For $a < c < b$, $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$
- If $f(x) \geq 0, \forall\ x \in [a, b]$ then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x), \forall\ x \in [a, b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If $m \leq f(x) \leq M, \forall\ x \in [a, b]$ then
  $m \cdot (b - a) \leq \int_a^b f(x)dx \leq M \cdot (b - a)$

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus (FToC) essentially gives us a rule to calculate any integral we encounter.

- **Definition:** Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Then the **antiderivative (indefinite integral)** of $f(x)$ is a function $F(x)$ that satisfies

$$F(x) = \int f(x)dx \Leftrightarrow F'(x) = f(x) .$$

- Notice that an antiderivative is **not** unique. If $F(x)$ is an antiderivative of $f(x)$ then so is $[F(x) + c]$ as $[F(x) + c]' = F'(x) = f(x)$.

Fundamental Theorem of Calculus

When a closed form of an antiderivative exists, we can use the FToC to evaluate definite integrals.

Fundamental Theorem of Calculus: let $f(x)$ be a continuous function on an interval $[a, b]$ and let $F(x)$ be an antiderivative of $f(x)$. Then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \ .$$

Fundamental Theorem of Calculus

Example: evaluate the following integral: $F(x) = \int x^5 dx$.

- Recall the definition of an antiderivative: $F(x)$ is an antiderivative of an integrable function $f(x)$ if

$$F(x) = \int f(x)dx \Leftrightarrow F'(x) = f(x) \ .$$

- We, thus, need to find such a function $F(x)$ whose derivative $F'(x)$ equals to $f(x) = x^5$.
  If $F(x) = \frac{x^6}{6} + c$ then $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x^6}{6} + c\right) = x^5 = f(x)$.
  Therefore, $F(x) = \int f(x)dx = \dfrac{x^6}{6} + c$.

Evaluating Definite Integrals

Evaluating definite integrals eventually boils down to finding antiderivatives. The process of finding an antiderivative is not as straightforward as the process of differentiation.

There are, however, certain rules and methods that facilitate finding an antiderivative.

- Anti-Power Rule: $\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + c$.
- Tabular cases:

$f(x)$	0	c	$sin(x)$	$cos(x)$	e^x	$\frac{1}{x}$	$\dots$
$F(x) + c$	0	cx	$-cos(x)$	$sin(x)$	e^x	$ln(x)$	$\dots$

Evaluating Definite Integrals

Example:

evaluate the following definite integral: $\int_2^5 \left(3x^2 + 2x + 1\right) dx$.

$$\int_2^5 \left(3x^2 + 2x + 1\right) dx = \int_2^5 3x^2 dx + \int_2^5 2x dx + \int_2^5 1 dx =$$

$$3 \cdot \int_2^5 x^2 dx + 2 \cdot \int_2^5 x dx + \int_2^5 1 dx =$$

$$3 \cdot \frac{1}{3}\, x^3\Big|_2^5 + 2 \cdot \frac{1}{2}\, x^2\Big|_2^5 + x\Big|_2^5 = (125 - 8) + (25 - 4) + (5 - 2) = 141$$

Integration By Parts

Integration By Parts is one of the methods of finding an antiderivative of $f(x)$. To compute the antiderivative the technique uses the product rule backwards.

A derivation of the method is given below:

- Let $f(x), g(x)$ be continuous, differentiable functions. Recall the definitions of antiderivatives:

$$F(x) = \int f(x)dx \Leftrightarrow \frac{d}{dx}F(x) = f(x)$$

$$G(x) = \int g(x)dx \Leftrightarrow \frac{d}{dx}G(x) = g(x)$$

Integration By Parts

- We are now interested in the expression for the derivative of the product of the antiderivatives:

$\frac{d}{dx}\left[F(x) \cdot G(x)\right] = \frac{d}{dx}F(x) \cdot G(x) + \frac{d}{dx}G(x) \cdot F(x) =$

$f(x) \cdot G(x) + g(x) \cdot F(x)$

- From the definition of an antiderivative

$$F(x) = \int f(x)dx \Leftrightarrow \frac{d}{dx}F(x) = f(x) \ .$$

Integrating the right-hand side of the equivalence over x yields

$$\int \left(\frac{d}{dx}F(x)\right) = \int f(x)dx = F(x) \ .$$

Integration By Parts

- Let's now integrate both parts of

  $\frac{d}{dx}[F(x)\cdot G(x)] = f(x)\cdot G(x) + g(x)\cdot F(x)$ over x.

  $$\Rightarrow \int \left(\frac{d}{dx}[F(x)\cdot G(x)]\right) dx = \int (f(x)\cdot G(x))\, dx + \int (g(x)\cdot F(x))\, dx$$

  $$\Rightarrow F(x)\cdot G(x) = \int f(x)\cdot G(x) dx + \int g(x)\cdot F(x) dx \ .$$

- The **Integration By Parts** rule:

  $$\boxed{\int g(x)\cdot F(x) dx = F(x)\cdot G(x) - \int f(x)\cdot G(x) dx}$$

Integration By Parts

- **Hint:** choose $F(x)$ and $g(x)$ so that $\int f(x) \cdot G(x)$ is easy to compute.
- $\int$ **byParts:**

$$\boxed{\int u(x) \cdot v'(x)dx = u(x)v(x) - \int u'(x) \cdot v(x)dx}$$

Integration By Parts

Example: derive the expression for $\int ln(1+x)dx$.

- $\int$ **byParts:** let $f(x), g(x)$ be continuous, differentiable functions, then

$$\int g(x) \cdot F(x)dx = F(x) \cdot G(x) - \int f(x) \cdot G(x)dx \ .$$

- Let $\int \underbrace{ln(1+x)}_{F(x)} \cdot \underbrace{1}_{g(x)} dx$, then

$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(ln(1+x)\right) = \frac{1}{1+x}$ and

$G(x) = \int g(x)dx = \int 1dx = x + C$.

If we let $C = 1$ then $G(x) = x + 1$.

Integration By Parts: Definite Integrals

For **definite integrals:**

$$\int_a^b [f(x) \cdot G(x) + g(x) \cdot F(x)]\, dx = \int_a^b \frac{d}{dx} [F(x) \cdot G(x)] =$$

$$F(x) \cdot G(x)\big|_a^b = F(b) \cdot G(b) - F(a) \cdot G(a)\ .$$

Therefore, for definite integrals the **Integration By Parts** rule:

$$\boxed{\int_a^b F(x) \cdot g(x) = F(b) \cdot G(b) - F(a) \cdot G(a) - \int_a^b f(x) \cdot G(x) dx}$$

Integration By Substitution

Integration By Substitution is another method of finding an antiderivative of $f(x)$. To compute the antiderivative the technique uses the chain rule backwards.

A derivation of the method is given below:

- Let $F(x) = \int f(x)dx$ and $x = g(u)$. Thus, $F(x) = F(g(u))$.
- We are now interested in the expression for the derivative of the obtained composite function $F(g(u))$ with respect to u.

Integration By Substitution

- $\frac{d}{du}[\ F(g(u))\ ] = F'(g(u)) \cdot g'(u) = f(g(u)) \cdot g'(u)$
- Integrating both parts of the equation above over u we obtain:

$$\int \frac{d}{du}[\ F(g(u))\ ]\, du = \int f(g(u)) \cdot g'(u) du\ .$$

Thus, $F(g(u)) = \int f(g(u)) \cdot g'(u) du$.

- **Integration By Substitution** rule: Let $f(x)$ be an integrable and $g(x)$ be invertible and continuously differentiable functions. The substitution $x = g(u)$ changes the integration variable from x to u as follows:

$$\int f(x) dx = \int f(g(u)) \cdot g'(u) du\ .$$

Integration By Substitution

Example: derive the expression for $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

- Let $\sqrt{x} = u$. Then $x = u^2$ and from $\frac{dx}{du} = 2u$ we obtain $dx = 2udu$.
- Substituting the newly derived terms into the original equation, we get:

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^u}{u} 2u\, du = 2 \cdot \int e^u du = 2 \cdot e^u + C = 2 \cdot e^{\sqrt{x}} + C \, .$$

Integration By Substitution: Definite Integrals

- For **definite integrals:** using the **Fundamental Theorem of Calculus** we obtain:

$$\int_{x=a}^{x=b} f(x)dx = F(x)\Big|_{x=a}^{x=b} = \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} f(g(u)) \cdot g'(u)du =$$

$$\int_{u=g^{-1}(a)}^{u=g^{-1}(b)} \frac{d}{du}\left[F(g(u))\right] du =$$

$$F\left(\ g(g^{-1}(b))\ \right) - F\left(\ g(g^{-1}(a))\ \right) = F(b) - F(a)\ .$$

- **Integration By Substitution** rule for definite integrals is given by:

$$\boxed{\int_{x=a}^{x=b} f(x)dx = \int_{u=g^{-1}(a)}^{u=g^{-1}(b)} f(g(u)) \cdot g'(u)du}$$

Taylor Series

- **Taylor's Formula:**

  The value of function $f(x)$ in the neighbourhood of point a can be expressed as:

  $$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}, \quad c \in [a, x] \ ,$$

  where the first n+1 term are referred to as the **Taylor approximation of order** n and the last term is the **Lagrange's form of the reminder** (or the **error term**).

Taylor Series

- The linear approximation of $f(x)$ about $x = a$ is:

  $f(x) \approx f(a) + f'(a)(x - a)$.

- The quadratic approximation of $f(x)$ around $x = a$ is:

  $f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$.

- The Taylor Approximation of order n of $f(x)$ around $x = a$ is:

  $f(x) \approx f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n$.

Taylor Series

- **Lagrange's Form of Remainder:**

  If $f(x)$ is $n+1$ times differentiable in an interval $[a, x]$, then the remainder can be written as:

  $R_{n+1}(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$ for some $c \in [a, x]$.

- **Error Estimation:**

  $|R_{n+1}(x)| \leq \frac{M}{(n+1)!} \cdot |x-a|^{n+1}$, where $|f^{(n+1)(x)}| \leq M$.

Taylor Series

- $f(0) = 1$
- $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$
- $f''(x) = -\frac{1}{2} \cdot \frac{1}{2}(1+x)^{-\frac{3}{2}} = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$
- $f'''(x) = -\frac{3}{2} \cdot (-\frac{1}{4})(1+x)^{-\frac{5}{2}} = \frac{3}{8}(1+x)^{-\frac{5}{2}} \Rightarrow f'''(0) = \frac{3}{8}$

Therefore,

$$f(x) \approx 1 + \frac{1}{1!}\frac{1}{2}x - \frac{1}{2!}\frac{1}{4}x^2 + \frac{1}{3!}\frac{3}{8}x^3 = 1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 \ .$$

Taylor Series

Graphically: