Linear Algebra

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Outline

Definition

An $n \times m$ matrix is a rectangular arrey with n rows and m columns:

$$A = (a_{ij})_{n \times m} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

a_{ij} denotes the element in the i^{th} row and the j^{th} column.
Remark: Elements of a matrix can belong to different sets, we however will concentrate on $A \in \mathbb{R}^{n \times m}$

Dimension of a Matrix

The **dimension** or the **order** of a matrix is defined in terms of the number of rows and the number of colums it contains.

$$A = A^{n \times m}$$

where n is the number of rows,
and m is the number of colums

Example

$$A = \begin{pmatrix} 1 & 2 & 5 & 8 \\ 9 & 4 & 2 & 0 \end{pmatrix}$$

The dimention of A is 2×4

Rank of a Matrix

Rank of a given matrix A is the number of it's linearly independent rows/columns.
We say that vectors are **linearly independent** if none of them can be represented as a linear combination of the others.

Example

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

It is easy to see that the third line is just a sum of the first two.
Therefore $\text{rank}(B)$ must be at most 2.
You can use the Gaussian elimination method to determine rank of a given matrix.

Remark: rank is defined not only fore square matrices.

Transpose of a Matrix

$$C = A' = A^\top \iff (c_{ij}) = (a_{ji})$$

Example 1

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^\top = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Example 2

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 9 & 2 & 1 \end{pmatrix} \Rightarrow A^\top = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 5 & 2 \\ 3 & 0 & 1 \end{pmatrix}$$

Types of Matrices

- Zero matrix

$$(a_{ij}),\ \forall i,j\ a_{ij} = 0$$

- Square matrix

$$A^{\mathbf{n}\times\mathbf{n}}$$

The following examples are all square matrices

- Identity matrix

$$I = (a_{ij}) \text{ such that } a_{ii} = 1 \text{ and } a_{ij} = 0\ \forall i \neq j$$

Example

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} ; I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Types of Matrices

- Symmetric matrix

$$A = (a_{ij}) \text{ such that } a_{ij} = a_{ji} \left[\Leftrightarrow A = A^\top \right]$$

Example 1

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & 7 \\ 5 & 7 & 3 \end{pmatrix}$$

Example 2

$$I$$

Types of Matrices

- Upper/lower triangular matrix
  - A square matrix is called *lower triangular* if all the entries above the main diagonal are zero.
  - A square matrix is called **upper triangular** if all the entries below the main diagonal are zero.
  - A matrix that is both upper and lower triangular is a **diagonal matrix**.

Example

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 3 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 8 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}, C = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- A **triangular matrix** is one that is either lower triangular or upper triangular.

Operations on Matrices

1. **Addition**

$$C = A + B$$
$$(c_{ij}) = (a_{ij} + b_{ij})$$

Example

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -9 \end{pmatrix},\ B = \begin{pmatrix} 3 & 1 \\ 2 & 13 \end{pmatrix} \Rightarrow C = A + B = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$$

2. **Scalar multiplication**

$$kA = k \cdot (a_{ij}) = (ka_{ij})$$

Example

$$2 \cdot \begin{pmatrix} 1 & 0 & 2 \\ 0 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & -10 & 2 \end{pmatrix}$$

Rules for Matrix Addition and Product by Scalars

- $(A + B) + C = A + (B + C)$
- $A + B = B + A$
- $A + 0 = A$
- $A + (-A) = 0$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(A + B) = \alpha A + \alpha B$

Operations on Matrices

3. **Scalar product of two vectors**

$$a = (a_1, a_2, ..., a_n)$$
$$b = (b_1, b_2, ..., b_n)$$
$$a \bullet b = < a, b >= \sum_{i=1}^{n} a_i \cdot b_i$$

Note that the vectors have to be of the same dimension!

Example

$$a = (1, 2, 3),\ b = (3, 2, 1)$$
$$\Rightarrow < a, b >= a \bullet b = ab^\top = 1 \times 3 + 2 \times 2 + 3 \times 1 = 10$$

Operations on Matrices

4. **Multiplication of matrices**

$$A = A^{n \times k},\ B = B^{k \times m}$$
$$A \cdot B = C^{n \times m} = (c_{ij})$$
$$c_{ij} = < a_{i*}, b_{*j} >$$

Where a_{i*} is the i^{th} row of matrix A and b_{*j} is the j^{th} column of B.
Note that the number of columns of A must be equal to the number of rows of B. Otherwise the product of the two matrices is **not defined**!

Example

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 12 & 3 \\ 11 & 4 \end{pmatrix}$$

Rules for Matrix Multiplication

- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- For a square matrix A, $IA = AI = A$

Note that for matrices it is usually the case that $AB \neq BA$!

Example

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & 5 \\ 1 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3 & 14 \\ -1 & 7 \end{pmatrix}; BA = \begin{pmatrix} 10 & -5 \\ 7 & 0 \end{pmatrix}$$

Rules for Transposition

5. We have already defined what a transpose matrix is, now we can look at the rules of transposition.

    - $(A')' = A$
    - $(A+B)' = A' + B'$
    - $(\alpha A)' = \alpha A'$
    - $(AB)' = B'A'$

Operations on Matrices

6. **Inverse**
   The inverse A^{-1} of a square matrix A is defined as a matrix that satisfies the following condition:

$$B = A^{-1} \iff AB = BA = I$$

Inverse of 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

$$\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}^{-1} = \frac{1}{6+4} \begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.4 \\ 0.1 & 0.2 \end{pmatrix}$$

Properties of the Inverse

If the relevant inverses exist, the following rules apply:

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(cA)^{-1} = \frac{1}{c}A^{-1}$
- $(A')^{-1} = (A^{-1})'$

Determinant

- A **dermenant** of 2×2 matrix is:

$$\det(A) = \left|A\right| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- For a general $n \times n$ matrix $A = (a_{ij})$, $det(A)$ is defined recursively. For any $i, j = 1, 2, ..., n$

$$\left|A\right| = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} = \sum_{k=1}^{n} a_{ik}A_{ik}$$

$$\left|A\right| = a_{1j}A_{1j} + a_{2j}A_{2j} + ... + a_{nj}A_{nj} = \sum_{k=1}^{n} a_{kj}A_{kj}$$

where A_{mk} is the product of $(-1)^{m+k}$ and the determinant of $(n-1) \times (n-1)$ matrix obtained by deleting the m^{th} row and the k^{th} column of the original matrix A.
Note that A_{mk} is a real number, not a matrix!

Rules for Determinants

- If all elements in a row/column of A are zero, then $\det(A) = 0$.
- $\left|A'\right| = \left|A\right|$.
- If two rows/columns of A are proportional, then $\det(A) = 0$.
- $\left|AB\right| = \left|A\right| \cdot \left|B\right|$.
- $\left|\alpha A\right| = \alpha^n \cdot \left|A\right|$, for $\alpha \in \mathbb{R}$.
- A has an inverse $\Longleftrightarrow \left|A\right| \neq 0$
- $\det(A) \neq 0 \Leftrightarrow \text{rank}(A) = n$

Examples of Applications in Economics

Example: Finance

Consider a financial market under certainty consisting of two payment streams and two payment dates:

1. security with payment stream $(120, 150)$
2. straight bond with maturity 2 years, principal 100 and yearly coupon of 6%

Prices at time zero for the avobe listed securities are 220 and 100 respectively.

Examples of Applications in Economics

Example: Finance

This market can be represented using matrix notation as follows:

$$X = \begin{pmatrix} 120 & 150 \\ 6 & 106 \end{pmatrix}, \pi = \begin{pmatrix} 220 \\ 100 \end{pmatrix}$$

where X is the payoff matrix and π is the price vector.

- The discount factors d_1 and d_2 can be found by solving $Xd = \pi$.
- The discount factors can be than used to price any future payment stream (x_1, x_2)

Examples of Applications in Economics

Other application you will see:

- Econometrics - OLS.
- Input-Output models.
- A convenient notation for budget constraints in microeconomic models.

Gaussian Elimination

Gaussian Elimination is a method used to determine a rank of a given matrix, to invert matrices, and to solve systems of linear equations.
Let us first consider the method in finding the inverce of a given matrix A.

- First construct the $n \times 2n$ matrix $(A|I)$ by writing down the n columns of A followed by the n columns of the identity matrix.
- By applying elementary row operations transform the matrix into an $n \times 2n$ matrix of a form $(I|B)$.
- If it is impossible to perform such row operations, then A is not invertable.
- Otherwise $B = A^{-1}$

Elementary row operations

Elementary row operations are:

- Multiplying/ dividing a row by scalar.
- Adding/subtracting one row to/from another.
- Switching between two rows.

Gaussian Elimination

Example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 2 & -6 \end{pmatrix}, A^{-1} = ?$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ -3 & 2 & -6 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & -6 & 3 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -6 & 3 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & -0.5 & \frac{1}{6} & -\frac{1}{6} \end{array}\right)$$

Exersice: verify that $AA^{-1} = A^{-1}A = I$

Gaussian Elimination

Having found the inverse of matrix A, one can easily solve the system of equations $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, by multiplying both sides by the inverse,

$$A^{-1}Ax = A^{-1}b$$

$$Ix = A^{-1}b \Rightarrow x = A^{-1}b$$

On the other hand, it is not always necessary to know what the inverse matrix is.
If the goal is to solve the system of equations, the method should be applied to the matrix $(A|b)$.

Gaussian Elimination

Example

We want to solve the following system of equations:

$$\begin{aligned} 2x_1 + 4x_3 &= 8 \\ 3x_2 + 5x_3 &= 0 \\ x_1 + 4x_3 &= 4 \end{aligned}$$

The matrix form of this problem is $Ax = b$, where

$$A = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & 5 \\ 1 & 0 & 4 \end{pmatrix}, b = \begin{pmatrix} 8 \\ 0 \\ 4 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Gaussian Elimination

Solution

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 0 & 4 & 8 \\ 0 & 3 & 5 & 0 \\ 1 & 0 & 4 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 3 & 5 & 0 \\ 1 & 0 & 4 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow x_1 = 4, x_2 = x_3 = 0$$

Gaussian Elimination

Using the method to find rank of a given matrix:
In order to find a rank of a given matrix, one has to find the echelon form of the matrix.
The number of non-zero rows in the echelon form of a matrix equals the rank of this matrix.
A matrix is in **echelon form** when it satisfies the following conditions:

- The first non-zero element in each row, called the leading entry, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

Gaussian Elimination

Example

What is the rank of the matrix $A = \begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 5 \\ 4 & 3 & 5 \end{pmatrix}$?

Solution

$$\begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 5 \\ 4 & 3 & 5 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 5 \\ 4 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

We have brought the matrix to it's echelon form. There are two independent rows, the rank of this matrix is 2.

Solution of $Ax = b$

We have seen that a system of linear equations can be represented in a matrix form, $AX = b$.
There are three possible cases regarding the number of solutions for each system of equations:

- there is no solution,
- there exists a unique solution,
- there are infinite many solutions.

Solution of $Ax = b$

Example: No solution

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \end{pmatrix}, b = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 6 \\ 4 & 8 & 8 & 10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 4 & 4 & 6 \\ 2 & 4 & 4 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 4 & 4 & 6 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$\Rightarrow 0 = -1$ which of course does not hold, thus there is no solution to this system of linear equations.

Solution of $Ax = b$

Example: Unique solution

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 8 & 8 \\ 4 & 4 & 8 \end{pmatrix}, b = \begin{pmatrix} 8 \\ 16 \\ 12 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 8 \\ 0 & 8 & 8 & 16 \\ 4 & 4 & 8 & 12 \end{array}\right) \overset{\frac{l_2}{2}}{\sim} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 8 \\ 0 & 4 & 4 & 8 \\ 4 & 4 & 8 & 12 \end{array}\right) \underset{l_3 - l_2}{\overset{l_1 - l_2}{\sim}} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 4 & 4 & 8 \\ 4 & 0 & 4 & 4 \end{array}\right)$$

$$\overset{l_3 - 2 \cdot l_1}{\sim} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 4 & 4 & 8 \\ 0 & 0 & 4 & 4 \end{array}\right) \underset{\frac{l_1}{2}}{\overset{l_2 - l_3}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 4 & 4 \end{array}\right) \underset{\frac{l_3}{4}}{\overset{\frac{l_3}{4}}{\sim}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow x_1 = 0, x_2 = 1, x_3 = 1$$

Solution of $Ax = b$

Example: Infinite many solutions

$$A = \begin{pmatrix} 2 & 4 & 4 \\ 4 & 8 & 4 \end{pmatrix}, b = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 6 \\ 4 & 8 & 4 & 10 \end{array}\right) \overset{\frac{l_3}{2}}{\sim} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 6 \\ 2 & 4 & 2 & 5 \end{array}\right) \overset{l_2 - l_1}{\sim} \left(\begin{array}{ccc|c} 0 & 0 & 2 & 1 \\ 2 & 4 & 2 & 5 \end{array}\right)$$

$$\underset{\frac{l_2}{2}}{\overset{\frac{l_1}{2}}{\sim}} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 0.5 \\ 1 & 2 & 1 & 2.5 \end{array}\right) \Rightarrow x_3 = 0.5, x_1 + 2x_2 + 0.5 = 2.5$$

$$\Rightarrow x_3 = 0.5, x_1 = 2 - 2x_2.$$

Denoting $x_2 = \alpha$ we see that any vector $x' = \begin{pmatrix} 2 - 2\alpha & \alpha & 0.5 \end{pmatrix}$ is a solution.

Solution of $Ax = b$

The following criteria can be used in order to determine how many solutions a system of linear equations $Ax = b$ has:

- If $rank(A) = rank(A|b) = n$ the system has an unique solution. (n is the dimention of vector x)
- If $rank(A) = rank(A|b) = k < n$ the system has infinite many solutions.
- If $rank(A) \neq rank(A|b)$ the system has no solution

Farkas-Stiemke Lemma

Let A be a $n \times m$ matrix. Then precisely one of the following is true:

1. There exists $x \in \mathbb{R}^m, x \gg 0$ such that $Ax = 0$.
2. There exists $y \in \mathbb{R}^n$ such that $A'y > 0$.

Eigenvalues and Eigenvectors

Def: A vector $x \neq 0$ is said to be an **eigenvector** of a matrix A if there exisits a constant λ such that $Ax = \lambda x$. The constant λ is in this case refered to as the **eigenvalue**.

Example

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}, \quad x = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ -12 \end{pmatrix} = 2 \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

Therefore the given vector x as an eigenvector of the given matrix A with the eigenvalue $\lambda = 2$.

Determination of Eigenvalues

According to the definition, $Ax = \lambda x$, therefore

$$Ax - \lambda x = 0 \Leftrightarrow Ax - \lambda Ix = 0 \Leftrightarrow \Big(A - \lambda I\Big)x = 0$$

As the eigenvector $x \neq 0$ this implies that the system $\Big(A - \lambda I\Big)x = 0$ has a non zero solution, which can only be if the matrix $\Big(A - \lambda I\Big)$ is not invertable (otherwise there would be a unique solution, and here we have the trivial zero solution plus the non zero eigenvector.) And we know that a matrix is not invertable, if its determinant is equal to zero.

Determination of Eigenvalues

Therefore, in order to find eigenvalues of a matrix one has to solve the following equation

$$det\Big(A - \lambda I\Big) = 0$$

Example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad A - I\lambda = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

$$det\Big(A - \lambda I\Big) = (1-\lambda)(1 - 2\lambda + \lambda^2 - 2)) = (1-\lambda)(\lambda^2 - 2\lambda - 1)$$

the equation $(1-\lambda)(\lambda^2 - 2\lambda - 1) = 0$ has three solutions,

$$\lambda_1 = 1, \lambda_2 = 1 + \sqrt{2}, \text{and} \lambda_3 = 1 - \sqrt{2}$$

Determination of Eigenvalues

Having eidentified the eigenvalues, we can now find the corresponding eugenvectors. For each λ_i we need to solve $\Big(A - \lambda I\Big)x = 0$

Example

For λ_1=1 we have

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow x = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \forall a \neq 0$$

We can see that

$$Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \lambda_1 x$$

Determination of Eigenvalues

Example continued

For $\lambda_2 = 1 + \sqrt{2}$ we have

$$\begin{pmatrix} -\sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 2 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{array}{l} x_1 = 0 \\ \sqrt{2}x_2 = 2x_3 \Leftrightarrow x_2 = \sqrt{2}x_3 \\ x_2 = \sqrt{2}x_3 \end{array}$$

$$\Rightarrow x = \begin{pmatrix} 0 \\ \sqrt{2}a \\ a \end{pmatrix}, \forall a \neq 0$$

We can also check that

$$Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{2}a \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}a + 2a \\ \sqrt{2}a + a \end{pmatrix} = \begin{pmatrix} 0(1+\sqrt{2}) \\ \sqrt{2}a(1+\sqrt{2}) \\ a(\sqrt{2}+1) \end{pmatrix}$$

Determination of Eigenvalues

Example continued

That is

$$Ax = \begin{pmatrix} 0(1+\sqrt{2}) \\ \sqrt{2}a(1+\sqrt{2}) \\ a(\sqrt{2}+1) \end{pmatrix} = (1+\sqrt{2}) \begin{pmatrix} 0 \\ \sqrt{2}a \\ a \end{pmatrix} = \lambda_2 x$$

In the same way, one can see that the eigenvalue λ_3 has eigenvectors of the form

$$x = \begin{pmatrix} 0 \\ -\sqrt{2}a \\ a \end{pmatrix}, \forall a \neq 0$$