Introduction to Set Theory

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Math Preparation Course

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Outline

Notation

- "$\in$" - is an element of / belongs to set
- "$\notin$" - is not an element of / does notbelong to
- "$\exists$" - there exists/ there is
- "$\forall$" - for all/ for every
- " $\vee$ "- or
- " $\wedge$ "- and
- Sets are usually denoted by capital letters (A, B, S).
- The elements of a set are denoted by small letters and listed in curly brackets ({...})

Notation

Examples:

- $a \in A$
- $d \notin A$
- $\exists x : x \in A \wedge x \in B$
- $\forall c \in C \ c = 2 \cdot n, \ n = 1, 2, 3, \ldots$

Different ways to define a set

1. Explicitely list the elements:

Example

- $A = \{1, 2, 6, 125\}$
- $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, ...\}$
- $D = \{a, b, c\}$

Note:

- The order of the elements is not important:

$$\{2, 1\} = \{1, 2\}$$

- Repitition of elements is irrelevant:

$$\{1, 1, 1, 1, 1\} = \{1\}$$

Different ways to define a set

2. Define a set using verbal description:

Examples

- B is the set of all even numbers between 4 and 25
- $\mathbb{R}$ is the set of all real numbers

3. Define a set using a mathematical rule/condition:

Examples

- $K = \{x \in \mathbb{Z} | -2 < x < 5\}$
- $A = \{x \in \mathbb{N} | x = 3 \cdot n - 1, n \in \mathbb{N}\}$

Important Sets of numbers

- The **Integer** numbers:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

- The **Natural** numbers:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, ...\}$$

or sometimes

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, ...\}$$

Remark: . There is no general agreement about whether to include 0 in the set of natural numbers

- The **Real** numbers :

$$\mathbb{R} = (-\infty, +\infty)$$

Important Sets of numbers

- The **n-dimentional real space** :

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) | x_i \in \mathbb{R}, i = 1, 2, ..., n\}$$

- The **Empty Set**

$$\emptyset = \{\}$$

  is the set that has no elements.

- The **Rational** numbers :

$$\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}, q \neq 0\}$$

Important Sets of numbers

- All the real numbers which are not rational are called **Irrational** numbers.
- **Intervals**

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

$$[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$$

$$[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leqslant b\}$$

Subsets

Subset:
Given two sets A and B, we say that A is a subset of B if every element of A is also an element of set B.

$$A \subset B \Leftrightarrow x \in A \Rightarrow x \in B$$

- B is called a **superset** of A
- One can also say tha A is contained in B
- Any set is a subset of itself, i.e. $A \subset A$
- If $A \subset B$ but is not equal to B, it is called a **proper subset** $A \subsetneq B$.

Operations on Sets

1 **Union of sets** $A \cup B$

$$A \cup B = \{x | x \in A \vee x \in B\}$$

Example:

$$\begin{array}{c} A = \{a, b, c, e\} \\ B = \{b, d\} \end{array} \Rightarrow A \cup B = \{a, b, c, d, e\}$$

2 **Intersection of sets** $A \cap B$

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

Example:

$$A \cap B = \{b\}$$

Two sets A and B are said to be **disjoint** when $A \cap B = \emptyset$.

Operations on Sets

Sometimes one is interested in an union or intersection of infinite number of sets.

Example

For $n = 1, 2, 3, \ldots$

$$I_n = \left[0, \frac{1}{n}\right]$$

$$I_1 = [0, 1], I_2 = \left[0, \frac{1}{2}\right], I_3 = \left[0, \frac{1}{3}\right] \ldots$$

$$\Rightarrow \bigcap_{n=1}^{\infty} I_n = \{0\}, \bigcup_{n=1}^{\infty} I_n = [0, 1]$$

Operations on Sets

3 **Cartesian product of sets:** $A \times B$

$$A \times B = \{(a, b) | a \in A, b \in B\}$$
$$A \times B \times C = \{(a, b, c) | a \in A, b \in B, c \in C\}$$

Example:

$$\begin{array}{l} A = \{1, 2\} \\ B = \{3, 4\} \end{array} \Rightarrow A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

Note: the order of the sets **is important!**

$$A \times B \neq B \times A$$

Operations on Sets

4 **Complement of a set A^C**
Let U be the universal set, and $A \subset U$. Than the compliment of A is the set of elements in U that are not in A, also denoted as A^C and A'.

$$A^C = A' = U/A = U - A$$

Example:

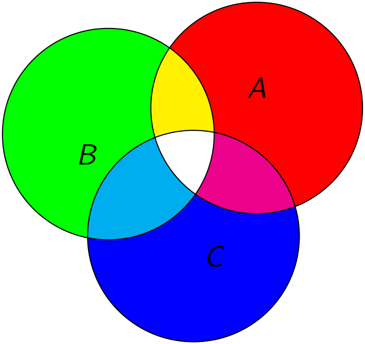
$$\begin{matrix} A = (1,2] \\ U = [0,2] \end{matrix} \Rightarrow A^C = [0,1]$$

Complement Laws

- $A' \cup A = U$
- $A' \cap A = \emptyset$
- $U' = \emptyset$
- $\emptyset' = U$
- $(A')' = A$
- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

Venn Diagram

Relationships between a small number of sets can be represented in a graphical way, refered to as the Venn diagram:



Example:

The white region: $A \cap B \cap C$; The green region is: $B/(A \cup C)$

Some more useful notations

Summation or product of a finite large or an infinite number of elements are denoted as follows:
Given an infinite series of numbers $a_1, a_2, a_3, ..$ we can denote the sum and the product of those numbers by:

- **Sum**

$$\sum_{i=1}^{\infty} a_i$$

- **Product**

$$\prod_{i=1}^{\infty} a_i$$

Note: of course the number of elements does not have to be infinite. We can replace ∞ by the desiered number of elements. We also note that even though it is not explicitly stated, one assumes that the index is in the set of natural numbers.