Tutorial 3

1 Optimization

Question 1:
Find the maximum or/and the minimum of the following functions. Describe the intervals of convexity and concavity.

a. $f(x) = 2 - x^2$

b. $f(x) = x^2 - 7x + 10$

c. $f(x) = x^3$

d. $f(x) = x^3 - 5x$

e. $f(x) = x^4 - 2x^2$

f. $f(x) = \dfrac{x^2 - 1}{x^2 - 4}$

Question 2:
Find the extreme values of the following functions and say whether they are local or absolute maxima or minima.

a. $f(z) = z^2 + 4z - 10$

b. $g(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x$

c. $F(y) = \frac{1}{4}y^4 + \frac{2}{3}y^3 + 3y^2 + 8$

Question 3:
Solve the following optimization problems:

a.
$$\max_{l,k} u(l,k) = l^{\frac{1}{3}} k^{\frac{2}{3}}$$
s.t.
$$l + k = 3$$

b.
$$\max_{x,y} u(x,y) = alnx + by$$
s.t.
$$x + y = c$$

c.
$$\max_{x_1,x_2} U = x_1 x_2 + 2x_1$$
s.t.
$$4x_1 + 2x_2 = 60$$

d.
$$\min_{x_1,x_2} w = p_1x_1 + p_2x_2$$
s.t.
$$z = kx_1^a x_2^b$$

e.
$$\min_{x,y} 7.5x + 3y$$
s.t.
$$6h_1 + 2h_2 \geq 10$$
$$7x + 3y \geq 10$$
$$6x + 2y \geq 20$$
$$10x + 5y \geq 20$$

f.
$$\max_{C_1,C_2} lnC_1 + \frac{1}{1+\rho} lnC_2$$
s.t.
$$C_1 + \frac{C_2}{1+r} = y_1 + \frac{y_2}{1+r}$$

Question 4:
Cournot Competition(simultaneous quantities setting in duopoly)
Consider a 2-firm economy with one good. The market demand is described by $q^D = 10-p$, where p is the market price of the good. The costs of producing one unit of good is 1$ for firm A and 2$ for firm B.

a. Represent the profit (revenue-costs)function for both firms.

b. Find the profit maximization production level for both firms.

c. Find the equilibrium production level and market price.

2 Probability Theoty

Question 1:
Assume that three mutually independent random variables X, Y, Z have the following means and sdandard deviations: $\mu_X = 5, \mu_Y = 3, \mu_Z = 2, \sigma_X = 0.25, \sigma_Y = 2, \sigma_Z = 1$.

Find:

a. $\mathbb{E}[4X^2 + Y - 8Z]$.

b. $\mathbb{E}[(Z - Y)^2]$.

c. $\mathbb{V}ar[(X + 4Y]$.

d. Show that the random variable $W = \frac{X - \mu_x}{\sigma_x}$ has zero mean and variance equal to 1.

Let $\{X_1, X_2, ..., X_n\}$ be i.i.d. random variables with mean μ and variance σ^2.

e. Find the mean and the standard deviation of $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$.

f. Find the expected value of $S = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{n}$ and of $\bar{S} = \frac{\sum_{i=1}^{n}(X_i - \mu)^2}{n-1}$

Question 2:
Let X and Y be two random variables.

a. Give a formal definition of covariance and of correlation coefficient between the two variables.

b. Show that if X and Y are linearly dependent, then $|\rho_{XY}| = 1$.

Question 3:
For given set of observations $(x_i, y_i), i = 1, 2, ..., n$ solve the following optimization problem:

$$\min_{a,b} \sum_{i=1}^{n}(y_i - ax_i - b)^2.$$

Show that $a^* = \rho_{XY} \frac{\sqrt{Var(X)}}{\sqrt{Var(Y)}}$.

Question 4:
Suppose you want to sell N_A units of an asset at time t_2 and choose to hedge by shorting futures contracts on N_F units of similar asset at time t_1. The hedge ratio is $h = \dfrac{N_F}{N_A}$. It can be shown that the optimal hedge ratio is $h^* = \rho\frac{\sigma_S}{\sigma_G}$, where ρ is the correlation between ΔG and ΔS, and σ denotes the standard deviation.

Futures price change (ΔG_i)	Fuel price change (ΔS_i)
0.021	0.029
0.035	0.020
-0.046	-0.044
0.001	0.008
0.044	0.026
-0.029	-0.019
-0.026	-0.010
-0.029	-0.007
0.048	0.043
-0.006	0.011

Assume that you plan to sell fuel and use futures on oil as hedging instrument. What should be your hedging ratio?

Question 5:
Aggregate supply and demand in Lukas model are given by the following system of equations:

$$p = \tfrac{1}{1+b}m + \tfrac{b}{1+b}\mathbb{E}[p]$$

$$y = \tfrac{b}{1+b}m - \tfrac{b}{1+b}\mathbb{E}[p]$$

where y denotes the output, m is the money supply, p is the price level abd b is a constant.

Find the equilibrium price level and output level (in terms of the rest of the parameters).